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The non-abelian, PST and supersymmetric formulations of Hull's notons

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Abstract: Chiral p-forms are, in fact, present in many supersymmetric and supergravity models in two, six and ten dimensions. In this work, the dual projection procedure, which is essentially equivalent to a canonical transformation, is used to diagonalize some theories in $D = 2$ (0-forms). The dual projection performed here provides an alternative way of gauging the chiral components without the necessity of constraints. It is shown that the non-mover field (the noton) initially introduced by Hull to cancel out the Siegel anomaly, has non-abelian, PST and supersymmetric formulations.

Keywords: Field Theories in Lower Dimensions, Chiral Lagrangians.

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1. Introduction

Chiral p-forms enter in the spectrum of type-IIB superstring and it can be shown that a string can couple directly to a chiral p-form [1]. A chiral p-form has a number of properties which are very peculiar. Although its excitations obey Bose statistics, it shares many of the features of fermionic fields like a field equation which is linear in derivatives; it leads to species doubling if one attempts to define it on a lattice [2] and it gives rise to gravitational anomalies [3].

We can describe chiral bosons by a p-form gauge potentials B_p. The B_p's curvatures $H_{p+1} = dB_p$ satisfy, as equations of motions, a Hodge (anti) self-duality condition in a space-time with dimension $D = 2(p+1)$. The values of p are restricted to even values by the self-consistency of such equation in space-times with Minkowskian signature $\eta_{ab} = (1, -1, -1, \ldots, -1)$. The restriction on p makes $D = 2, 6, 10, \ldots,$ be the relevant dimensions of the chiral boson theories.

Chiral bosons are important in superstring and supergravity theories, and more recently M-theory. Two-dimensional chiral bosons (scalar) are basic ingredients in string theory. The six-dimensional ones belong to the supergravity and tensor multiplets in $N = 1, D = 6$ supergravity theories. They are necessary to complete the $N = 2, D = 6$ supermultiplet of the M-theory five-brane. Finally a ten-dimensional chiral bosons appears in IIB, $D = 10$ supergravity.

In this work we study the dual projection technique introduced by Wotzasek et al. in [4, 5, 6]. This technique, strictly related to canonical transformation [7], has been presented in the study of electromagnetic duality groups. The difference between both concepts is that the dual projection is performed at the level of the actions while the canonical transformation is at the hamiltonian level. However, in this work we will consider only first-order actions. Furthermore, the equivalence between the dual projection and the canonical transformation becomes manifest [7]. The most useful and interesting point in this dual projection procedure is that it is not based on evidently even-dimensional concepts and may be extended to the odd-dimensional situation. But we will not analyze this feature here.

This work is motivated by the fact that coupling chiral fields to external gravitational fields reveals the presence of notons [8, 9]. Noton is a non-mover field at the classical level, carrying the representation of the Siegel symmetry [10], that acquires dynamics upon quantization. At the quantum level, it was shown [11], that its dynamics is fully responsible for the Siegel anomaly.

The Wess-Zumino-Witten-Novikov (WZWN) [12, 13] model is a conformal field theory that has been used in the past to reproduce various two-dimensional systems like Toda field theories, black holes and others. Recently it was carried out the connection of this model to a combination of three-dimensional topological BF and Chern-Simons gauge theories defined on a manifold with boundaries using a direct application of non-abelian T-duality on the WZWN non-linear sigma model [14].

In this work we discuss the dual projection of non-abelian chiral fields coupled to external gravitational backgrounds which is a result from the diagonalization of the first-order form of an action that describes the chiral WZWN model.

The Lorentz covariant approach to describe chiral bosons, proposed by Pasti, Sorokin and Tonin (PST) [15], has been studied in [16, 17, 18]. The quantum behavior of the PST models in the framework of the algebraic renormalization [19] was analyzed in [20]. Differently of [21] it introduces only one scalar auxiliary field, but in a non-polynomial way.

The PST procedure has important roles on the construction of a covariant effective action for the M-theory five-brane [17], covariant actions for supersymmetric chiral bosons [18] and to rederive the gravitational anomaly for the chiral bosons [16].

Here we disclose the presence of the noton in four models using the dual projection. In one we corroborate, in a clearer manner (in our point of view), the result found in [22]. Hence, we propose three new formulations for the Hull noton: two different Lorentz invariant versions of PST Siegel's chiral boson and a supersymmetric formulation.

The sequence of the paper is: in the next section we review the dual projection applied to the abelian chiral model and to the Principal Chiral Model as examples. In section 3 we carry out the dual projection of the chiral WZWN model. The analysis of two different versions of PST $D = 2$ chiral bosons and the supersymmetric one ([23]) are analyzed in section 4. Finally, the conclusion and perspectives are discussed in the last section.

2. The dual projection: a review

The separation of a scalar field into its chiral components has been introduced by Mandelstam [24] in its seminal paper on $2D$ bosonization. The chiral splitting for the non-abelian side has been studied by Polyakov and Wiegman [25]. However, the abelian limit of Polyakov-Wiegman decomposition does not coincide with Mandelstam's chiral decomposition. The chiral separation in [24] is based on a first-order theory while that of [25] is second order. Mandelstam's chiral decomposition scheme can be obtained as a constraint over the theory requiring the complete separability of the original action.

The dual projection in $D = 2(2p + 1)$ dimensions leads to a diagonal form of the action. Here, differently from [26], the two pieces manifest completely unlike features: while one piece is chiral, responsible for the dynamic sector of the theory, the other carry the algebraic component, as shown in [11]. We select here only two examples, among others, in the literature where the dual projection succeed.

2.1 The Siegel chiral boson

In this section we will make a brief review of the main results obtained in [11]. We begin with the Siegel original classical lagrangian density for a chiral scalar field [10],

$$\mathcal{L} = \partial_+ \phi \, \partial_- \phi - \lambda_{++} (\partial_- \phi)^2 = \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi \,, \tag{2.1}$$

where the metric

$$g^{++} = 0 \,, \qquad g^{+-} = 1 \,, \qquad g^{--} = -2\lambda_{++} \,, \tag{2.2}$$

and the lagrangian (2.1) describes a lefton [27]. A lefton (or a righton) is a particle which, besides to carry the dynamics of the theory, it is liable for the symmetry too. This characterizes exactly a Siegel mode. It is different from a Floreanini-Jackiw's (FJ's) mode [28]. Even if the FJ particle be a left mover, it is only responsible for the dynamics. Hence it is not a lefton.

The symmetry content of the theory is well described by the Siegel algebra, a truncate diffeomorphism that disappears at the quantum level. Hence (2.1) is invariant under Siegel gauge symmetry which is an invariance under the combined coordinate transformation and a Weyl rescaling of the form

$$x^- \longrightarrow \tilde{x}^- = x^- - \epsilon^- \,, \qquad \text{and} \qquad \delta g_{\alpha\beta} = -g_{\alpha\beta} \, \partial_- \epsilon^- \,. \tag{2.3}$$

The fields ϕ and λ_{++} transform under (2.3) as follows:

$$\begin{aligned} \delta\phi &= \epsilon^- \partial_- \phi \,, \\ \delta\lambda_{++} &= -\partial_+ \epsilon^- + \epsilon^- \partial_- \lambda_{++} - \partial_- \epsilon^- \lambda_{++} \,, \end{aligned} \tag{2.4}$$

and ϕ is invariant under the global axial transformation

$$\phi \longrightarrow \tilde{\phi} = \phi + \bar{\phi} \,. \tag{2.5}$$

It is well known that fixing the value of the multiplier as $\lambda_{++} = 1$ in (2.1) we can obtain the FJ model.

We will now begin to introduce the dual projection. This is done introducing a dynamical redefinition in the phase space of the model. Using (2.1) in Lorentz coordinates we can obtain the canonical momentum as

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} - \lambda_{++} \left(\dot{\phi} - \phi' \right) \tag{2.6}$$

and so we have that

$$\dot{\phi} = \frac{\pi - \lambda_{++}\,\phi'}{1 - \lambda_{++}}\,. \tag{2.7}$$

After a little algebra, substituting (2.6) and (2.7) into (2.1), the lagrangian in the first-order form reads

$$\mathcal{L} = \pi\dot{\phi} - \frac{\phi'^2}{2} - \frac{1}{2}\frac{(\pi - \lambda_{++}\,\phi')^2}{1 - \lambda_{++}} - \frac{\lambda_{++}}{2}\phi'^2\,. \tag{2.8}$$

As we said above, we have to fix the value of the multiplier as $\lambda_{++} \to 1$ to get the FJ form. This value of λ_{++} promotes a reduction of the phase space of the model to [29]

$$\pi \longrightarrow \phi'\,, \tag{2.9}$$

and consequently the third term in (2.8) reduces to zero as $\lambda_{++} \to 1$. Therefore the dynamics of the system will be described by a FJ action. The above behavior in (2.9) suggests the following canonical transformations:

$$\phi = \varphi + \sigma\,, \qquad \text{and} \qquad \pi = \varphi' - \sigma'\,, \tag{2.10}$$

and we stress that these fields are independent as they originate from completely different actions. After substituting (2.10) into (2.8) to perform the dual projection we find a diagonalized lagrangian,

$$\mathcal{L} = \varphi'\dot{\varphi} - \varphi'^2 - \sigma'\dot{\sigma} - \eta_+\sigma'^2\,, \tag{2.11}$$

where

$$\eta_+ = \frac{1 + \lambda_{++}}{1 - \lambda_{++}}\,.$$

The effect of dual projection procedure into the first-order Siegel theory, eq. (2.8) was the creation of two different internal spaces leading to the Z_2 group of dualities (a discrete group with two elements) [4, 5] and the other is the diffeomorphism group of transformations. Clearly we can see that the chirality/duality group and the symmetry group are in different sectors. The first is obviously a FJ mode and the other is a noton mode. This result is complementary to the established knowledge, where the FJ action is interpreted as a gauge fixed Siegel action [10]. Under this point of view, we look at the gauge fixing process (in eq. (2.1)) as the condition that sets the noton field to vanish. It can be proved [11] that this noton is totally responsible for the symmetries, both classically and quantically. We finally can say that the importance of the inclusion of a normalized external noton — the Hull mechanism — is, conveniently, to cancel the Siegel anomaly [8].

2.2 The principal chiral model

As a non-abelian example, let us next extend the separability condition, due to the dual projection, discussed above to the non-abelian bosons following only the main steps given in [26].

The most obvious choice would be to consider an action given by a bilinear gradient of a matrix-valued field g taking values on some compact Lie group G, which would be the natural extension of the free scalar abelian field. This is the action for the Principal Chiral Model (PCM) that reads

$$\mathcal{S}_{\rm PCM}(g) = \frac{1}{2}\int d^2x\, {\rm tr}\,(\partial_\mu g\, \partial^\mu \tilde{g})\,. \tag{2.12}$$

Here $g : R^{1,1} \to G$ is a map from the two-dimensional Minkowski space-time to G and let us write for simplicity $\tilde{g} = g^{-1}$. This action, however, puts some difficulties. First of all, by examining its field equation we learn that, it does not represent a free field. The jacobian of the field redefinition does not involve a time-derivative and can be reabsorbed in the normalization of the partition function. Let us write the PCM in its first-order form as

$$\mathcal{S}_{\rm PCM}(g,P) = \int d^2x\, {\rm tr}\,(\partial_\tau g P) + \frac{1}{2}\int d^2x\, {\rm tr}\,(PgPg) + \frac{1}{2}\int d^2x\, {\rm tr}\,(\tilde{g}\partial_\sigma g \tilde{g}\partial_\sigma g) \tag{2.13}$$

and redefine the fields g and P, which is an auxiliary field, through non-abelian canonical transformations

$$g = AB\,, \qquad \text{and} \qquad P = \varepsilon\left(\tilde{B}\partial_\sigma \tilde{A} - \partial_\sigma \tilde{B}\tilde{A}\right), \tag{2.14}$$

where, remembering our notation, we have that: $\tilde{A}, \tilde{B} = A^{-1}, B^{-1}$, respectively. The action for the PCM now reads

$$\mathcal{S}_{\rm PCM}\,(A,B) = \mathcal{S}_\varepsilon\,(A) + \mathcal{S}_{-\varepsilon}\,(B) + \varepsilon\int d^2x\, {\rm tr}\left[\tilde{A}\,(\partial_\tau A \partial_\sigma B - \partial_\sigma A \partial_\tau B)\,\tilde{B}\right], \tag{2.15}$$

where

$$\mathcal{S}_\varepsilon\,(A) = \int d^2x\, {\rm tr}\left(\varepsilon \partial_\sigma \tilde{A}\partial_\tau A - \partial_\sigma \tilde{A}\partial_\sigma A\right), \tag{2.16}$$

and $\mathcal{S}_{-\varepsilon}(B)$ is analogous. We see that due to the non-abelian nature of the fields, the cross-term cannot be eliminated, so that a complete separation cannot be achieved. This fact should be expected. In the canonical approach, we have for their field equations the pair

$$\partial_\tau I = \partial_\sigma J\,, \qquad \text{and} \qquad \partial_\tau J = \partial_\sigma I - [I,J]\,, \tag{2.17}$$

where $I = \partial_\tau g\, \tilde{g}$ and $J = \partial_\sigma g\, \tilde{g}$. The second equation is a sort of Bianchi identity, which is the integrability condition for the existence of g. Looking at this pair of equations one can appreciate that chirality is not well defined in this model. However, this picture changes drastically with the inclusion of the Wess-Zumino topological term, i.e., when we consider the WZWN model [12, 13]. The first equation in (2.17) changes to $\partial_\tau I = \partial_\sigma J + \rho[I,J]$, with $\rho \in Z$, producing a more symmetric set of equations. In particular for $\rho = \pm 1$ it is known that the currents above describe two independent affine Lie algebras. One expects then that with the introduction of the topological term, one would be able to obtain an identical mixing term, such that in the total action they could cancel each other. With the topological term

$$\Gamma_{WZ}(g) = \frac{1}{3}\int d^3x \epsilon^{ijk}\, {\rm tr}\,[\tilde{g}\partial_i g\, \tilde{g}\partial_j g \tilde{g}\partial_k g]\,, \tag{2.18}$$

we have now the complete WZWN model defined on the group manifold $\mathcal{M}_G$, based on the Lie algebra G, where $g \in \mathcal{M}_G$ and $\mathcal{M}$ is a three-dimensional ball whose the boundary is the two-dimensional surface $\partial\mathcal{M}$ [14].

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Using the first of the field redefinitions (2.14), it is a lengthy but otherwise straightforward algebra [26] to show that

$$\Gamma_{WZ}(A,B) = \Gamma_{WZ}(A) + \Gamma_{WZ}(B) + \int d^2x\,\mathrm{tr}\left[\tilde{A}\left(\partial_\tau A \partial_\sigma B - \partial_\sigma A \partial_\tau B\right)\tilde{B}\right]. \tag{2.19}$$

Next, we can bring results (2.15) and (2.19) into the WZWN action [12, 13], which is described by

$$\mathcal{S}_{WZWN}(g) = \frac{1}{\lambda^2}\mathcal{S}_{\mathrm{PCM}}(g) + \frac{n}{4\pi}\Gamma_{WZ}(g)\,. \tag{2.20}$$

We mention the appearance of an extra parameter, both in the action and in the canonical formalism, playing the role of coupling constant. In terms of the chiral variables, the WZWN model reads

$$\begin{aligned}\mathcal{S}_{WZWN}(A,B) = &\left[\frac{1}{\lambda^2}\mathcal{S}_{\varepsilon}(A) + \frac{n}{4\pi}\Gamma_{WZ}(A)\right] + \left[\frac{1}{\lambda^2}\mathcal{S}_{-\varepsilon}(B) + \frac{n}{4\pi}\Gamma_{WZ}(B)\right] + \\ &+ \left(\frac{\varepsilon}{\lambda^2} + \frac{n}{4\pi}\right)\int d^2x\,\mathrm{tr}\left[\tilde{A}\left(\partial_\tau A \partial_\sigma B - \partial_\sigma A \partial_\tau B\right)\tilde{B}\right].\end{aligned} \tag{2.21}$$

We can appreciate that the separability condition is only achieved at the critical points, as expected. But also that our choice of ε is now dependent on which of the critical points we choose: $4\pi\varepsilon = -\lambda^2 n$. The result is the non-abelian version of the dual projection, and corresponds to the sum of two lagrangians describing non-abelian chiral bosons of opposite chiralities, each one having the form proposed by Sonnenschein [30]. We will see that a change of the critical point automatically switches the chirality of A and B by changing the sign of ε. Indeed, in order to obtain separability, we must have either

$$\frac{\lambda^2 n}{4\pi} = -\varepsilon = 1 \tag{2.22}$$

or

$$\frac{\lambda^2 n}{4\pi} = -\varepsilon = -1\,. \tag{2.23}$$

In the first case we find the set of chiral equations as

$$\partial_x\left(\tilde{A}\partial_+ A\right) = 0\,, \qquad \text{and} \qquad \partial_-\left(\tilde{B}\partial_x B\right) = 0$$

whose solution reads, respectively,

$$A = A_-(x^-)h_A(t)\,, \qquad \text{and} \qquad B = h_B(t)B_+(x^+)\,.$$

In the second case, the chiral equations are

$$\partial_x\left(\tilde{A}\partial_- A\right) = 0\,, \qquad \text{and} \qquad \partial_+\left(\tilde{B}\partial_x B\right) = 0$$

and the solutions read

$$A = A_+(x^+)h_A(t)\,, \qquad \text{and} \qquad B = h_B(t)B_-(x^-)\,.$$

The arbitrary functions of time $h_A(t)$ and $h_B(t)$ represent, in fact, the zero modes of the solutions of the chiral equations. Note that in both cases the general solution for $g(x^+, x^-)$ is given as

$$g = A_\varepsilon(x^\varepsilon) h_A(t) h_B(t) B_{-\varepsilon}(x^{-\varepsilon}) \,.$$

As in the abelian case, the constraint $h_A(t) = h_B^{-1}(t)$ becomes necessary in order that $g = AB$ satisfies the equation of motion for the WZWN model. This constraint is the only memory left for the chiral bosons stating that they belong to the same non-chiral field. More details about the dual projection of PCM and the coupling with gravity can be seen in [26].

3. The new formulations of Hull's noton

There are indications that a deeper understanding of such issues as string dynamics and fractional quantum Hall effect phenomenology can be achieved by treating the chiral sectors in a more independent way. However, coupling chiral fields to external gauge and gravitational fields is problematic. As we said above (it was discussed in [26]) the coupling of chiral (abelian) fields to external gravitational backgrounds can be achieved by diagonalization (dual projection) of the first-order form of a covariant scalar action. The theory reduces then to a sum of a left and a right FJ's actions [28], circumventing the problems caused by the lack of manifest Lorentz invariance.

In this section we intend to supply the literature with new formulations of Hull's noton: the chiral WZWN model (using the dual projection), with two different versions of PST formulations of the chiral bosons and with the supersymmetric case. We hope that the use of the dual projection formalism and consequently the disclosure of a Hull's noton mode inside these models may help to gain a new insight into the structures of these theories. The redefinition of the fields in the first-order form of the action naturally reveals the two-dimensional internal structure hidden into the theory.

Note that in the last subsection, what we showed was the dual projection applied in a non-chiral model. Now the interesting touch is to see what will be disclosed in a well defined chiral model.

3.1 The chiral WZWN model

We know that the action for the chiral WZWN model [12, 13] for a particle which moves to the left, the so-called left mover is given by

$$S(g) = \int d^2x \, \mathrm{tr} \, (\partial_+ g \, \partial_- \tilde{g} + \lambda \partial_- g \, \partial_- \tilde{g}) + \Gamma_{WZ}(g) \,, \tag{3.1}$$

where, for convenience we write $\lambda = \lambda_{++}$ and $g \in G$ was defined in the last section. This action can be seen as the WZWN action immersed in a gravitational background, with a characteristic truncated metric tensor:

$$S(g) = \frac{1}{2} \int d^2x \sqrt{-\eta_+} \eta_+^{\mu\nu} \, \mathrm{tr} \, (\partial_\mu g \, \partial_\nu \tilde{g}) + \Gamma_{WZ}(g) \tag{3.2}$$

with $\eta^{+} = \det(\eta^{+}_{\mu\nu})$ and

$$\frac{1}{2}\sqrt{-\eta_{+}}\,\eta_{+}^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \lambda \end{pmatrix}, \tag{3.3}$$

where λ transforms like in the abelian case and g transforms as a scalar.

In Lorentz coordinates we can write (3.1) as

$$S(g) = \int d^2x \,\mathrm{tr}\left[\frac{1}{2}(1+\lambda)\dot{g}\dot{\tilde{g}} - \frac{1}{2}(1-\lambda)g'\tilde{g}' - \frac{\lambda}{2}(\dot{g}\tilde{g}' + g'\dot{\tilde{g}})\right] + \Gamma_{WZ}(g)\,. \tag{3.4}$$

This action can be written in its Faddeev-Jackiw's first-order form making the following transformation which introduces an auxiliary field $\mathcal{P}$:

$$\frac{1}{2}\dot{g}\,\dot{\tilde{g}} \longrightarrow \mathcal{P}\dot{g} + \frac{1}{2}\mathcal{P}g\mathcal{P}g\,,$$

where $\mathcal{P} = \dot{\tilde{g}}$.

Now we can write that

$$S(g) = \int d^2x \,\mathrm{tr}\left[(1+\lambda)\mathcal{P}\dot{g} + \frac{1}{2}(1+\lambda)\mathcal{P}g\mathcal{P}g - \frac{1}{2}(1-\lambda)g'\tilde{g}' - \frac{\lambda}{2}\left(\dot{g}\tilde{g}' + g'\dot{\tilde{g}}\right)\right] + \Gamma_{WZ}(g)\,. \tag{3.5}$$

For convenience, let us promote the following transformation of variables:

$$\mathcal{P} = \frac{1}{1+\lambda}\mathcal{P}\,,$$

and substituting in (3.5) we have that,

$$S(g) = \int d^2x \,\mathrm{tr}\left[\mathcal{P}\dot{g} + \frac{1}{2}\frac{1}{(1+\lambda)}\mathcal{P}g\mathcal{P}g - \frac{1}{2}(1-\lambda)g'\tilde{g}' - \frac{\lambda}{2}\left(\dot{g}\tilde{g}' + g'\dot{\tilde{g}}\right)\right] + \Gamma_{WZ}(g)\,. \tag{3.6}$$

Making again another transformation, i.e.

$$\mathcal{P} \longrightarrow \mathcal{P} + \lambda\dot{\tilde{g}}$$

hence,

$$S(g) = \int d^2x \,\mathrm{tr}\left[\mathcal{P}\dot{g} + \frac{1}{2}\frac{1}{1+\lambda}(\mathcal{P}g\mathcal{P}g - 2\lambda\mathcal{P}g') - \frac{1}{2}\frac{1}{(1+\lambda)}g'\tilde{g}'\right] + \Gamma_{WZ}(g)\,. \tag{3.7}$$

Let us redefine the fields g and $\mathcal{P}$ through the following canonical transformation:

$$g = Nh \qquad \Longrightarrow \qquad g' = N'h + Nh'\,, \tag{3.8}$$

where N, as before, will play the role of a Hull's non-mover field, the noton.

Using (2.9) to justify a first choice for $\mathcal{P}$, let us construct it with two numerical (or not) coefficients which will be determined in a further analysis,

$$\mathcal{P} = a\tilde{h}\tilde{N}' + b\tilde{h}'\tilde{N}\,, \tag{3.9}$$

but note the non-abelian feature of this different construction. An important observation is that the coefficients a and b incorporate the roles of the separability points described in the last section. We will show below that this form of the field $\mathcal{P}$ is the only possible general form in order to promote the dual projection.

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As we mentioned, it can be shown [26] that the dual projection of the Wess-Zumino term, using the redefinition (3.8) has the form

$$\Gamma_{WZ}(N,h) = \Gamma_{WZ}(N) + \Gamma_{WZ}(h) + \int d^2x \, \mathrm{tr}\left(\tilde{N}N'\dot{h}\tilde{h} - \tilde{N}\dot{N}h'\tilde{h}\right). \tag{3.10}$$

We can see that the dual projection of (3.10) brings an extra term that can not be split, so this term has to be eliminated. To perform this elimination it is easy to see that the cast of the field $\mathcal{P}$ has to have some term proportional to the extra term in (3.10). Other forms of such field do not afford the elimination of this term, as we said before.

The canonical transformations (3.8) and (3.9) lead us to an action with fields taking values in the internal space. Substituting the redefinitions (3.8), (3.9) and (3.10) into (3.7) we can show that,

$$\begin{aligned} S(N,h) &= \Gamma_{WZ}(N) + \Gamma_{WZ}(h) + \\ &+ \int d^2x \, \mathrm{tr}\Big\{a\tilde{N}'\dot{N} + b\tilde{h}'\dot{h} - (a-1)\tilde{N}\dot{N}\dot{h}\tilde{h} - (b+1)\tilde{N}\dot{N}h'\tilde{h} - \\ &\qquad - \frac{1}{1+\lambda}\left(\frac{a^2}{2} + \lambda a + \frac{1}{2}\right)\tilde{N}'N' - \frac{1}{1+\lambda}\left(\frac{b^2}{2} + \lambda b + \frac{1}{2}\right)\tilde{h}'h' + \\ &\qquad + 2(ab+1)\tilde{N}'Nh\tilde{h}'\Big\}. \end{aligned} \tag{3.11}$$

To eliminate the extra terms we have to find a and b solving the following very simple system:

$$a - 1 = 0\,, \qquad \text{and} \qquad b + 1 = 0\,, \tag{3.12}$$

which solution is

$$a = 1\,, \qquad \text{and} \qquad b = -1 \qquad \Longrightarrow \qquad ab = -1\,. \tag{3.13}$$

Instead of applying directly these solutions, let us promote a short analysis making $a = -b = \epsilon$. With this new form for the solution, the canonical transformations are

$$g = Nh\,, \qquad \text{and} \qquad p = \epsilon\tilde{h}\tilde{N}' - \epsilon\tilde{h}'\tilde{N}\,, \tag{3.14}$$

and substituting it in (3.11) we have the action given by

$$\begin{aligned} S(N,h) &= \int d^2x \, \mathrm{tr}\left[\epsilon\dot{N}\tilde{N}' - \frac{\epsilon^2 + 2\lambda\epsilon + 1}{2(1+\lambda)}N'\tilde{N}'\right] + \Gamma_{WZ}(N) - \\ &\quad - \int d^2x \, \mathrm{tr}\left[\epsilon\dot{h}\tilde{h}' - \frac{\epsilon^2 - 2\lambda\epsilon + 1}{2(1+\lambda)}h'\tilde{h}'\right] + \Gamma_{WZ}(h)\,, \end{aligned} \tag{3.15}$$

and finally we can say that

$$\begin{aligned} S(N,h)_{\epsilon=1} &= \int d^2x \, \mathrm{tr}\left(\dot{N}\tilde{N}' + N'\tilde{N}'\right) + \Gamma_{WZ}(N) - \int d^2x \, \mathrm{tr}\left(\dot{h}\tilde{h}' + \eta h'\tilde{h}'\right) + \Gamma_{WZ}(h)\,, \\ S(N,h)_{\epsilon=-1} &= -\int d^2x \, \mathrm{tr}\left(\dot{N}\tilde{N}' + \eta N'\tilde{N}'\right) + \Gamma_{WZ}(N) - \\ &\quad - \int d^2x \, \mathrm{tr}\left(\dot{h}\tilde{h}' + h'\tilde{h}'\right) + \Gamma_{WZ}(h)\,, \end{aligned} \tag{3.16}$$

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where

$$\eta = \frac{1-\lambda}{1+\lambda}\,.$$

Since we are working in $D = 2$ dimensions, i.e. 0-form ($D = 2(2p+1)$), each reduced phase space carries the representation for half the number of degrees of freedom of the original system. This is a feature different from $D = 4p$, where the duality symmetric systems maintains the phase space structure intact [6]. For the interested reader, an analysis of the WZWN duality groups is depicted in [31].

We see in (3.16) that in accordance with the value of ϵ, N and h change roles. For $\epsilon = 1$, N is the chiral field and h is the non-abelian Hull's noton. For $\epsilon = -1$, vice versa. This shows a certain dependence of the dual projection on a certain parameter, but it is immaterial since what we want to show is the presence, in the chiral WZWN, of a chiral mode and mainly the presence of a non-abelian Hull's noton.

This result corroborates the one found in [22]. However, there, the noton was found using the soldering technique [32] of two non-abelian chiral bosons, showing a destructive interference. Our work shows a different way to obtain the same result. Besides, we explain the result in [22] in a clearer way since now we can see both chiral particles and realize what is happening. As the WZWN model has a chiral mode and an algebraic mode, in the fusion (soldering) of opposite non-abelian chiral models, the opposite chiral particles interfere destructively disappearing from the spectrum. Only the Hull noton survives, which as having no dynamics, does not interact destructively. In other words, we show that the Hull noton was already there, inside the chiral WZWN model and consequently it is not just a product of soldering both opposite models as can be understood from [22].

3.2 The PST self-dual formulation

It can be shown [23] that one of the possibilities to recover the manifest Lorentz invariance of the FJ model, as in the case of Maxwell theory, is introducing an unit-norm time-like auxiliary vector field $u_m(x)$ in the FJ action, and to write the action in the form

$$S = \int d^2x \left(\partial_+ \phi\, \partial_- \phi - u^m \mathcal{F}_m u^n \mathcal{F}_n\right), \tag{3.17}$$

where $\mathcal{F}_m = \partial_m \phi - \epsilon_{mn}\partial^n \phi$. Because of its properties, u_m contains only one independent component. It was proved in [23] that (3.17) reduces to the Siegel action [10] and also that u_m is the gradient of a scalar. It will be used below.

In Minkowski space we can write the lagrangian density of (3.17) as

$$\mathcal{L} = \dot{\phi}^2 - \phi'^2 - \left(u^0 + u^1\right)^2 \left(\dot{\phi} + \phi'\right)^2 \tag{3.18}$$

and following the procedure to promote the dual projection in first-order actions we have, after a little algebra, that

$$\mathcal{L} = \pi\, \dot{\phi} - \frac{[\pi + 2(u^0+u^1)^2\, \phi']^2}{4[1-(u^0+u^1)^2]} - \left[1 + \left(u^0+u^1\right)^2\right] \phi'^2\,, \tag{3.19}$$

where π is the canonical momentum.

Now, as before, we can use the convenient canonical transformations,

$$\phi = \pm\frac{1}{\sqrt{2}}(\rho + \sigma)\,, \qquad \text{and} \qquad \pi = \pm\sqrt{2}(\rho' - \sigma')\,. \tag{3.20}$$

where one of these fields will represent the FJ particle and the other, of course, the noton, as we will see.

Substituting these transformations in (3.19) and after an algebraic work we have an action in terms of σ and ρ, which take values in the internal space,

$$\mathcal{L} = \left(-\sigma'\dot{\sigma} - \sigma'^2\right) + \left(\rho'\dot{\rho} - \eta_1\rho'^2\right), \tag{3.21}$$

where

$$\eta_1 = \frac{1 + (u^0 + u^1)^2}{1 - (u^0 + u^1)^2}\,.$$

As we can easily see the σ field describes a FJ chiral boson that now, at first sight, carries the dynamics of the system and the Z_2 duality group. However, in [6] it was shown that in all even dimensions, both SO(2) and Z_2 duality symmetric groups could co-exist. As we have analyzed in the anterior section, the ρ field is responsible for the algebra [23] of the system. Hence, ρ is the PST Hull's noton formulation.

The other possibility (see [23]), which is more appropriate from the quantum point of view, is to construct a Lorentz covariant action like the following:

$$S = \int d^2x \left(\partial_+\phi\,\partial_-\phi + \frac{1}{u^2}u^m\mathcal{F}_m u^n\mathcal{F}_n - \epsilon^{mn}\,u_m\partial_n B\right), \tag{3.22}$$

where $B(x)$ is an auxiliary scalar field. After eliminating B through the field equations and taking the values for u_m, the lagrangian density of the above action can be written as

$$\mathcal{L} = \partial_+\phi\,\partial_-\phi - \frac{\partial_+\hat{\varphi}}{\partial_-\hat{\varphi}}(\partial_-\phi)^2\,, \tag{3.23}$$

where $\hat{\varphi}$ is an another auxiliary field that helps in the solution of the equation for u_m, i.e. $u_m(x) = \partial_m\hat{\varphi}(x)$. Hence, $u_m(x)$ can be seen as the gradient of a scalar $\hat{\varphi}(x)$.

In this case it is easy to see that the eq. (3.23) has the same form as the Siegel chiral boson, where

$$\lambda \longrightarrow \frac{\partial_+\hat{\varphi}}{\partial_-\hat{\varphi}}$$

and with a straightforward association we already have the solution given in [11]

$$\mathcal{L} = \rho'\dot{\rho} - \rho'^2 - \sigma'\dot{\sigma} - \eta_2\sigma'^2\,, \tag{3.24}$$

where we have again σ as the noton and

$$\eta_2 = \frac{1 - \frac{\partial_+\hat{\varphi}}{\partial_-\hat{\varphi}}}{1 + \frac{\partial_+\hat{\varphi}}{\partial_-\hat{\varphi}}}\,.$$

In the face of this result we can make an analogous analysis for this second possibility of Lorentz invariance restoration as we made it for the first one and stress that the dynamics/$(Z_2 + \mathrm{SO}(2))$ duality groups and algebra group are carried by ρ and σ, respectively. We see clearly that Hull's noton is present whatever the way we formulate the Siegel chiral boson.

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3.3 The supersymmetric formulation

The superfield generalization of (3.22) is

$$S = \int d^2x d\theta^+ \left(D_+\Phi\, \partial_-\Phi - \frac{D_+\Lambda}{\partial_-\Lambda}(\partial_-\Phi)^2 \right), \tag{3.25}$$

where we are considering the bosonic superfields

$$\Phi(x^-, x^+, \theta^+) = \phi(x) + i\theta^+\psi_+(x)$$

and

$$\Lambda(x^-, x^+, \theta^+) = \hat{\varphi}(x) + i\theta^+\chi_+(x)\,,$$

which obey the conventional transformation laws under global shifts

$$\delta\theta^+ = \epsilon^+\,, \qquad \delta x^+ = i\theta^+\epsilon^+\,, \qquad \text{and} \qquad \delta x^- = 0$$

in $n = (1, 0)$ flat superspace and where

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta^+} + i\theta^+\partial_+\,, \\ D_+^2 &= i\partial_+ \end{aligned} \tag{3.26}$$

is the supercovariant derivative.

After making the relevant substitutions described above in (3.25) we can write

$$S = \int d^2x d\theta^+ \left\{ \frac{\partial\Phi}{\partial\theta^+}\partial_-\Phi - \frac{\partial\Lambda}{\partial\theta^+}\frac{(\partial_-\Phi)^2}{\partial_-\Lambda} + i\theta^+ \left[\partial_+\Phi\partial_-\Phi - \frac{\partial_+\Lambda}{\partial_-\Lambda}(\partial_-\Phi)^2 \right] \right\}. \tag{3.27}$$

Using the well known properties of the supersymmetric integrals we finally have that,

$$\mathcal{L} = \partial_+\Phi\partial_-\Phi - \frac{\partial_+\Lambda}{\partial_-\Lambda}(\partial_-\Phi)^2\,, \tag{3.28}$$

where a hidden global complex number is of no consequence here since it does not affect the equations of motion and may be ignored. With this form, we can make in (3.28) the same association that we made in (3.23). So, we can write, considering that now we are in a superspace that,

$$\mathcal{L} = \Sigma'\dot{\Sigma} - \Sigma'^2 - \Gamma'\dot{\Gamma} - \eta_3\Gamma'^2\,, \tag{3.29}$$

where

$$\eta_3 = \frac{1 - \frac{\partial_+\Lambda}{\partial_-\Lambda}}{1 + \frac{\partial_+\Lambda}{\partial_-\Lambda}}\,.$$

The superfield Σ represents the super-FJ chiral boson in the superspace formulation and Γ is the supersymmetric Hull's noton, the supernoton.[1]

[1] A very interesting study of the duality groups in supersymmetric models can be seen in [33].

4. Conclusion

We have diagonalized the non-abelian chiral field, disclosing a two dimension internal structure for some compact Lie group G. This result was expected based on the results found in [11] where it was shown that notons, being not dynamicals, couples to the gravitational backgrounds but not to the electromagnetic field. Therefore, if a gauge coupling is introduced before dual projection, it will be completely decoupled by the dual projection procedure. Since the analysis was always effected for first-order systems, an equivalence between the lagrangian and hamiltonian approaches permitted us to use the concept of canonical transformations. In other words we can say that the dual projection demanded a change of variables which was, in the phase space, a canonical transformation.

In [23] it was proposed two ways to restore the manifest Lorentz invariance of the FJ theory. The first was introducing an unit-norm time-like auxiliary vector field in the FJ action and constructing an action equivalent to the Siegel model. The other is constructing an action in the light of the Maxwell manifestly Lorentz invariant duality symmetric action proposed in [23], which is shown to be also Siegel's equivalent on the mass shell. We show that in both cases the noton is present. Our last result concerns the superfield generalization of the above action and a supersymmetric formulation of the noton, the supernoton has been introduced.

As a straightforward perspective we can diagonalize superiors orders of chiral p-forms and the investigation of the dual projection of superfield supergravity generalization of the duality symmetric models is in course.

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